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n-Torsion of Brauer groups as relative Brauer groups of abelian extensions

Cristian D. Popescu [a,*,1], Jack Sonn [b], Adrian R. Wadsworth [a]

[a] *University of California, San Diego, Department of Mathematics, 9500 Gilman Drive, La Jolla, CA 92093-0112, USA*

[b] *Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel*

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Abstract

It is now known [H. Kisilevsky, J. Sonn, Abelian extensions of global fields with constant local degrees, Math. Res. Lett. 13 (4) (2006) 599–607; C.D. Popescu, Torsion subgroups of Brauer groups and extensions of constant local degree for global function fields, J. Number Theory 115 (2005) 27–44] that if F is a global field, then the n-torsion subgroup ${}_n\mathrm{Br}(F)$ of its Brauer group $\mathrm{Br}(F)$ equals the relative Brauer group $\mathrm{Br}(L_n/F)$ of an abelian extension L_n/F, for all $n \in \mathbb{Z}_{\geqslant 1}$. We conjecture that this property characterizes the global fields within the class of infinite fields which are finitely generated over their prime fields. In the first part of this paper, we make a first step towards proving this conjecture. Namely, we show that if F is a non-global infinite field, which is finitely generated over its prime field and $\ell \neq \mathrm{char}(F)$ is a prime number such that $\mu_{\ell^2} \subseteq F^\times$, then there does not exist an abelian extension L/F such that ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$. The second and third parts of this paper are concerned with a close analysis of the link between the hypothesis $\mu_{\ell^2} \subseteq F^\times$ and the existence of an abelian extension L/F such that ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$, in the case where F is a Henselian valued field.

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* Corresponding author.

E-mail addresses: cpopescu@math.ucsd.edu (C.D. Popescu), sonn@math.technion.ac.il (J. Sonn), arwadsworth@ucsd.edu (A.R. Wadsworth).

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0. Introduction

This paper is concerned with the study of certain Galois theoretic properties of the n-torsion subgroup ${}_n\mathrm{Br}(F)$ of the Brauer group $\mathrm{Br}(F)$ of a field F. More precisely, in [AS], the authors raise the question whether ${}_n\mathrm{Br}(F)$ is equal to the relative Brauer group $\mathrm{Br}(L/F) := \ker(\mathrm{Br}(F) \to \mathrm{Br}(L))$ of a separable algebraic extension L/F. They showed that the answer to this question is negative in general, giving an example with F a power series field over a local field, and the present paper provides a systematic way of producing such examples (see Corollary 1.6 and Proposition 2.2 and Corollary 2.3 below). On the other hand, somewhat surprisingly, the answer turns out to be positive for global fields F:

Theorem 0.1. [KS2] *If F is a global field* (*i.e. a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(T)$, where T is a variable and p is prime*)*, then for all integers $n \geqslant 2$ there exists an abelian extension* (*necessarily of infinite degree*) *L_n/F, such that ${}_n\mathrm{Br}(F) = \mathrm{Br}(L_n/F)$.*

In [KS1], this result was proved for number fields F under certain restrictions on the pair (n, F) (in particular for all n when $F = \mathbb{Q}$). In [Po], the result was proved for all global function fields F of characteristic p when n is a power of p. In [KS2] the result was proved in full for all number fields and all global function fields F of characteristic p with $(n, p) = 1$. This, together with the result in [Po] gives Theorem 0.1.

We conjecture that the conclusion in the theorem above characterizes global fields within the class of infinite fields which are finitely generated over their prime field. Although we are currently unable to fully confirm this conjecture, in Section 1 of the present paper we make encouraging steps towards doing so by proving the following (see Corollary 1.6 below).

Theorem 0.2. *Let F be an infinite field which is finitely generated over its prime field and is not a global field and let ℓ be a prime number, $\ell \neq \mathrm{char}(F)$, such that $\mu_{\ell^2} \subseteq F^\times$. Then, there does not exist an abelian extension L/F such that ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$.*

As usual, μ_n denotes the group of roots of unity of order dividing n, for all $n \in \mathbb{Z}_{\geqslant 1}$. Although at present we are unable to remove the condition $\mu_{\ell^2} \subseteq F^\times$ in the theorem above, in Sections 2–3 below we study more closely the link between this condition and the existence of L/F algebraic such that ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$, in the case where F is a Henselian valued field (and therefore not finitely generated over its prime field) of residue characteristic different from ℓ. Examples of Henselian valued fields include (but are not restricted to) all fields which are complete with respect to a discrete valuation, in particular all local fields and all fields of iterated power series in finitely many variables over any field. It turns out that in this case, if the ℓ-rank of the value group Γ_F of F is at least 2 (i.e. $\dim_{\mathbb{Z}/\ell\mathbb{Z}}(\Gamma_F/\ell\Gamma_F) \geqslant 2$) and the residue field of F is finite, then the existence of an extension L/F as above is equivalent to $\mu_{\ell^2} \nsubseteq F^\times$ (see Proposition 2.2 below). Obviously, if the ℓ-rank of Γ_F is equal to 1, then this is not an equivalence, as the example of local fields shows (see the introduction to Section 2 below). The case where the residue field of F is infinite is more complicated. We give a partial answer to the question of the existence of L/F in this case (see Proposition 2.1), which allows us to construct explicit examples of extensions L/F in the particular case where $F = \mathbb{Q}((t))$, for all primes ℓ (see Section 3).

In what follows we use standard notation. In particular, if F is a field endowed with a valuation v, then we denote by V_v, M_v and $\overline{F}_v$ the corresponding valuation ring, maximal ideal and residue field, respectively. If $x \in V_v$, we denote by $\bar{x}$ its image in the residue field $\overline{F}_v$. If the valuation v is discrete of rank 1, then $\widehat{F}_v$ denotes the completion of F in the v-adic topology. For the basic properties of central simple algebras and Brauer groups used in this paper, the reader may consult [Pi,Se]. We will use repeatedly the fact that if K/F is a cyclic field extension, $a \in F^\times$ and σ generates the Galois group $\mathcal{G}(K/F)$, then the cyclic F-algebra $(K/F, \sigma, a)$ is split if and only if a belongs to the image of the norm map $\mathrm{N}_{K/F} : K \longrightarrow F$ (see [Pi, §15.1, Lemma]).

1. Finitely generated fields

Our main goal in this section is to prove Theorem 0.2 announced in the introduction (see Corollary 1.6 below).

Lemma 1.1. *Let ℓ be a prime number. Let $F \subseteq L$ be fields such that L is Galois over F with Galois group $\mathcal{G}(L/F)$ abelian of exponent ℓ. Let v be a discrete valuation on F with $\mathrm{char}(\overline{F}_v) \neq \ell$, and let w be any extension of v to L. Then,*

(i) *The ramification index $e_{w/v} = 1$ or ℓ. If $e_{w/v} = \ell$, then there is a field K with $F \subseteq K \subseteq L$, $[K : F] = \ell$, and K is totally ramified over F with respect to w. If, further, $\mu_\ell \subseteq F^\times$, then there is $\pi \in F$ with $v(\pi) = 1$ and $K = F(\sqrt[\ell]{\pi})$.*

(ii) *$\overline{L}_w$ is a Galois extension of $\overline{F}_v$ with $\mathcal{G}(\overline{L}_w/\overline{F}_v)$ abelian of exponent dividing ℓ. Furthermore, there is a field M with $F \subseteq M \subseteq L$, $\overline{M}_w = \overline{L}_w$, with v inert and unramified in M (so $\mathcal{G}(M/F) \cong \mathcal{G}(\overline{L}_w/\overline{F}_v)$, canonically).*

(iii) *Suppose $\mu_\ell \subseteq F^\times$. If $\widetilde{a} \in \overline{F}_v$ with $\sqrt[\ell]{\widetilde{a}} \in \overline{L}_w$, then there is $a \in F^\times$ with $v(a) = 0$ such that $\bar{a} = \widetilde{a}$ and $\sqrt[\ell]{a} \in L$.*

Proof. Since L is a direct limit of finite degree extensions which satisfy the same hypotheses as L, it suffices to prove the lemma when $[L : F] < \infty$. Assume this. Let $G = \mathcal{G}(L/F)$, and let D and I be the decomposition group and the inertia subgroups of G relative to w.

(i) Since $[L : F]$ is a power of ℓ, it is prime to $\mathrm{char}(\overline{F}_v)$. Hence, $I \cong \mathbb{Z}/e_{w/v}\mathbb{Z}$, so $e_{w/v} = |I| = \exp(I) \mid \ell$. Suppose $e_{w/v} = \ell$, and let I' be a complement of I in the elementary abelian ℓ-group G. Let K be the fixed field of I', and let w_0 be the restriction of w to K. Then, $[K : F] = |G/I'| = |I| = \ell$. The inertia group of w_0/v is $II'/I' = G/I'$, so $e_{w_0/v} = |II'/I'| = [K : F]$. Thus, w_0 is totally ramified over v, and is the unique extension of v to K. Now, suppose $\mu_\ell \subseteq F^\times$. By Kummer theory, $K = F(\sqrt[\ell]{c})$ for some $c \in F^\times$. Take any $\pi_v \in F^\times$ with $v(\pi_v) = 1$. Suppose $\ell \mid v(c)$, say $v(c) = k\ell$; then $K = F(\sqrt[\ell]{d})$, where $d = c(\pi_v^{-k})^\ell$, with $v(d) = 0$. If $\bar{d} \notin \overline{F}_v^{\times\ell}$, then $\sqrt[\ell]{\bar{d}} \in \overline{K}_{w_0} \setminus \overline{F}_v$, a contradiction to K being totally ramified over F. But, if $\bar{d} \in \overline{F}_v^{\times\ell}$, then for the ring $R = V_v[\sqrt[\ell]{d}\,]$ we have $R/M_vR \cong \overline{F}_v[x]/(x^\ell - \bar{d})$, which is a direct sum of ℓ fields by the Chinese Remainder Theorem. Therefore, the integral closure T of V_v in K, which is integral over R, must contain a distinct maximal ideal lying over each of the ℓ maximal ideals of R. Localizing T with respect to each maximal ideal gives a distinct discrete valuation ring of K extending V_v. This contradicts the uniqueness of w_0 extending v to K. These contradictions force $\ell \nmid v(c)$. Hence, writing $1 = i\ell + jv(c)$, we can set $\pi = c^j(\pi_v^i)^\ell$. Then, $v(\pi) = 1$ and $F(\sqrt[\ell]{\pi}) = F(\sqrt[\ell]{c^j}) = F(\sqrt[\ell]{c}) = K$, as $\ell \nmid j$.

(ii) Let J/I be a complement of D/I in the elementary abelian ℓ-group G/I, and let M be the fixed field of J, and w_1 the restriction of w to M. The decomposition group of w_1 over v is $DJ/J = G/J$. Hence, v is inert in M. The inertia group of w_1 over v is $IJ/J = (1)$; so, w_1

is unramified over v. Let N be the fixed field of I, which is the inertia field of w over v. The decomposition group of $w|_N$ over w_1 is $(D/I)\cap(J/I)=(1)$. Hence, w_1 is totally decomposed in N; so $\overline{M}_{w_1}=\overline{N}_{w|_N}=\overline{L}_w$. We have $\mathcal{G}(M/F)\cong\mathcal{G}(\overline{M}_{w_1}/\overline{F}_v)=\mathcal{G}(\overline{L}_w/\overline{F}_v)$.

(iii) Suppose $\mu_\ell\subseteq F^\times$ and $\widetilde{a}\in\overline{F}_v$ with $\sqrt[\ell]{\widetilde{a}}\in\overline{L}_w$. We may assume that $\sqrt[\ell]{\widetilde{a}}\notin\overline{F}_v$. From the isomorphism of Galois groups in (ii), there is a field M_0 with $F\subseteq M_0\subseteq M$ and $\overline{M_0}=\overline{F}_v(\sqrt[\ell]{\widetilde{a}})$ in $\overline{M}_{w_1}$. We have $[M_0:F]=[\overline{F}_v(\sqrt[\ell]{\widetilde{a}}):\overline{F}_v]=\ell$. By Kummer theory, $M_0=F(\sqrt[\ell]{b})$ for some $b\in F^\times$. We have $\ell\mid v(b)$ since M_0 is unramified over F; hence, we may assume $v(b)=0$. If $\bar{b}\in\overline{F}_v^{\ell}$, then v would be totally decomposed in M_0, as we saw in the proof of (i). This cannot happen since v is inert in M, and so in M_0. Hence, $\bar{b}\notin\overline{F}_v^{\times\ell}$. As $\overline{F}_v(\sqrt[\ell]{\bar{b}})\subseteq\overline{M_0}_w$ and $[\overline{F}_v(\sqrt[\ell]{\bar{b}}):\overline{F}_v]=\ell=[\overline{M_0}_w:\overline{F}_v]$, we have $\overline{F}_v(\sqrt[\ell]{\bar{b}})=\overline{M_0}_w=\overline{F}_v(\sqrt[\ell]{\widetilde{a}})$. By Kummer theory, $\widetilde{a}=\bar{b}\bar{c}^\ell$ for some $c\in F^\times$ with $v(c)=0$. Set $a=bc^\ell$. Then, $v(a)=0$ and $\bar{a}=\bar{b}\bar{c}^\ell=\widetilde{a}$ and $F(\sqrt[\ell]{a})=F(\sqrt[\ell]{b})=M_0\subseteq L$. □

Proposition 1.2. *Let ℓ be a prime number, and let F be a field with $\mu_{\ell^2}\subseteq F^\times$. Suppose that L is an abelian Galois extension field of F with* $\exp(\mathcal{G}(L/F))=\ell$ *and* ${}_\ell\mathrm{Br}(F)=\mathrm{Br}(L/F)$. *Let v be a discrete valuation of F with* $\mathrm{char}(\overline{F}_v)\neq\ell$, *and let w be any extension of v to L. Then,*

(i) *If w is ramified over v, then $\overline{L}_w=\overline{F}_v$.*
(ii) *If w is unramified over v, then $\overline{F}_v\subseteq(\overline{L}_w)^\ell$ and* ${}_{\ell^2}\mathrm{Br}(\overline{F}_v)={}_\ell\mathrm{Br}(\overline{F}_v)$.

Proof. (i) Since w is ramified over v, there is $\pi\in F^\times$ with $v(\pi)=1$ and $\sqrt[\ell]{\pi}\in L$ (see Lemma 1.1(i)). Since $\overline{L}_w$ is an ℓ-Kummer extension of $\overline{F}_v$, if $\overline{F}_v\subsetneqq\overline{L}_w$, there is $\widetilde{a}\in\overline{F}_v\setminus(\overline{F}_v)^\ell$ with $\sqrt[\ell]{\widetilde{a}}\in\overline{L}_w$. By Lemma 1.1(iii), there is $a\in F^\times$ with $v(a)=0$, $\bar{a}=\widetilde{a}$, and $\sqrt[\ell]{a}\in L$. Let A be the symbol algebra $(a,\pi/F)_{\ell^2}$. Then $\exp(A)=\ell^2$ since $A^{\otimes\ell}\sim(a,\pi/F)_\ell$ and $(a,\pi/F)_\ell$ is nonsplit, as $\pi\notin\mathrm{im}N_{F(\sqrt[\ell]{a})/F}$ (see the last paragraph of the introduction). (Since v is inert and unramified in $F(\sqrt[\ell]{a})/F$, we have $v(\mathrm{im}(N_{F(\sqrt[\ell]{a})/F}))\subseteq\ell\mathbb{Z}$.) But A is split by its maximal subfield $F(\sqrt[\ell]{a},\sqrt[\ell]{\pi})\subseteq L$, so L splits A. This contradicts $\mathrm{Br}(L/F)={}_\ell\mathrm{Br}(F)$.

(ii) Suppose w is unramified over v. Let $\pi\in F^\times$ with $v(\pi)=1$. Take any $b\in F^\times$ with $v(b)=0$, and let $B=(b,\pi/F)_\ell$. By hypothesis, $(b,\pi/L)_\ell=B\otimes_F L$ is split. Since $w(\pi)=v(\pi)=1$, this implies that $b\in\mathrm{im}(N_{L(\sqrt[\ell]{\pi})/L})$, so $\bar{b}\in(\overline{L}_w)^\ell$. Thus, $\overline{F}_v\subseteq(\overline{L}_w)^\ell$, as asserted. Since $\mu_{\ell^2}\subseteq\overline{F}_{v^\times}$, by the Merkurjev–Suslin Theorem (see [Sr, §8]), ${}_{\ell^2}\mathrm{Br}(\overline{F}_v)$ is generated by symbol algebras of degree ℓ^2. Let $\widetilde{C}=(\widetilde{c},\widetilde{d}/\overline{F}_v)_{\ell^2}$, and suppose $\exp(\widetilde{C})=\ell^2$ in $\mathrm{Br}(\overline{F}_v)$. Since we just proved that $\sqrt[\ell]{\widetilde{c}},\sqrt[\ell]{\widetilde{d}}\in\overline{L}_w$, by Lemma 1.1(iii) there are $c,d\in F^\times$ with $v(c)=v(d)=0$, $\bar{c}=\widetilde{c}$, $\bar{d}=\widetilde{d}$, and $\sqrt[\ell]{c},\sqrt[\ell]{d}\in L$. Let $C=(c,d/F)_{\ell^2}$. Let $p:=\mathrm{char}(\overline{F}_v)$. In what follows, if G is an abelian group, G' denotes the "prime to p-part" of G, i.e. $G':=G\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]$. Witt's Theorem (see [Se, Chapter XII, Theorem 2 and Example 3]) gives an explicit group isomorphism

$$\mathrm{Br}(\widehat{F}_v)'\cong\mathrm{Br}(\overline{F}_v)'\oplus\mathrm{Hom}_c(G_{\overline{F}_v},\mathbb{Q}/\mathbb{Z})',$$

where $\mathrm{Hom}_c(G_{\overline{F}_v},\mathbb{Q}/\mathbb{Z})$ denotes continuous homomorphisms (i.e. homomorphisms with open kernel in the Krull topology on the absolute Galois group $G_{\overline{F}_v}$ of $\overline{F}_v$). In the composite homomorphism

$$\mathrm{Br}(F)'\longrightarrow\mathrm{Br}(\widehat{F}_v)'\longrightarrow\mathrm{Br}(\overline{F}_v)'\oplus\mathrm{Hom}(G_{\overline{F}_v},\mathbb{Q}/\mathbb{Z})',$$

$[C]$ maps to $[\widetilde{C}]\in\mathrm{Br}(\overline{F}_v)$. Hence, $\ell^2=\exp(\widetilde{C})\mid\exp(C)\mid\ell^2$, so equality holds throughout. However, L splits C, since it contains the maximal subfield $F(\sqrt[\ell]{c},\sqrt[\ell]{d})$ of C. This contradicts

$\mathrm{Br}(L/F) = {}_\ell\mathrm{Br}(F)$. Hence, every symbol algebra of $\overline{F}_v$ of degree ℓ^2 has exponent at most ℓ. So, ${}_{\ell^2}\mathrm{Br}(\overline{F}_v) = {}_\ell\mathrm{Br}(\overline{F}_v)$. □

We remind the reader of the following definition (see [FJ] for more details).

Definition 1.3. A field F is called *Hilbertian* if for every irreducible polynomial $f(X,Y) \in F[X,Y]$ there exist infinitely many $x_0 \in F$, such that the specialization $f(x_0, Y)$ is irreducible in $F[Y]$.

It is well known that global fields are Hilbertian and that if F is a finite extension of a rational function field $K(X)$ over an arbitrary field K, then F is Hilbertian (see [FJ, p. 155]). In particular, any infinite field which is finitely generated over its prime field is Hilbertian. For the following, see [FSS, proof of Theorem 2.6] as well.

Lemma 1.4. *Let k be a Hilbertian field, let F be a finite separable extension of $k(t)$, where t is transcendental over k, and let K be a cyclic Galois extension of F. Then, there is a discrete valuation v of F with an extension w to K such that $\overline{K}_w$ is cyclic Galois over $\overline{F}_v$ and $[\overline{K}_w : \overline{F}_v] = [K : F]$. Also, $\overline{F}_v$ is a finite degree extension of k.*

Proof. Since K is separable over $k(t)$, there is $\alpha \in K$ with $K = k(t)(\alpha)$. We can adjust α if necessary to assure that α is integral over $k[t]$. Let $f = f(t,x) \in k[t][x]$ be the minimal polynomial of α over $k(t)$. Because k is Hilbertian, there is $a \in k$ such that $f_a = f(a,x)$ is irreducible in $k[x]$. Let v be the $(t-a)$-adic valuation on $k(t)$, which has residue field $\overline{k(t)}_v = k$. If w is any extension of v to K, then the integrality of α over the valuation ring of v implies that $w(\alpha) \geqslant 0$. The image $\overline{\alpha}$ of α in $\overline{K}_w$ satisfies $f_a(\overline{\alpha}) = \overline{f(\alpha)} = 0$. Hence, by the irreducibility of f_a, we have $[\overline{K}_w : \overline{k(t)}_v] \geqslant \deg(f_a) = \deg(f) = [K : k(t)]$. Therefore, v is inert and unramified in K. Consequently, for the (unique) extension of v to F, again denoted v, we have v is inert and unramified in K. Since K is Galois over F, K_w is Galois over F_v and $\mathcal{G}(\overline{K}_w/\overline{F}_v) \cong \mathcal{G}(K/F)$, as desired. □

Theorem 1.5. *Let F be a field which is not a finite field nor a global field, but which is finitely generated over its prime field. Let ℓ be a prime number with $\mu_\ell \subseteq F^\times$ (so $\mathrm{char}(F) \neq \ell$). Let $\mathcal{N} = \mathrm{im}(N_{F(\mu_{\ell^2})/F})/F^{\times\ell}$. Suppose that there is an abelian Galois algebraic extension L of F with ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$. Let $\mathcal{K} = (L^{\times\ell} \cap F^\times)/F^{\times\ell}$. Then, $\mathcal{N} \cap \mathcal{K} = (1)$.*

Proof. Let L_1 be the maximal ℓ-primary subextension of F in L. Then, $\mathrm{Br}(L_1/F)$ is the ℓ-primary torsion subgroup of $\mathrm{Br}(L/F)$. Thus, by replacing L by L_1, we may assume that $\mathcal{G}(L/F)$ is an ℓ-primary abelian group. (This replacement does not change $\mathcal{K}$.) Because F is finitely generated over its prime field, it is separably generated (though not algebraic) over the prime field (see [Mat, §27.E, Corollary to Lemma 2, p. 194]). Therefore there is a subfield $k \subseteq F$ and an element $t \in F$ such that t is transcendental over k and F is a finite separable extension of $k(t)$. Since k is finitely generated over a global field, k is Hilbertian. We claim that $\mathcal{G}(L/F)$ is actually an ℓ-torsion group. For, if not, there is a field K, with $F \subseteq K \subseteq L$ and K cyclic Galois over F with $[K : F] = \ell^2$. Choose valuations v for F and w for K as in Lemma 1.4. Let A be the cyclic algebra $(K/F, \sigma, \pi_v)$, where $\pi_v \in F$ is a uniformizer for v and σ is a generator of $\mathcal{G}(K/F)$. Then, $\exp(A) = \ell^2$ because $v(N_{K/F}(K^\times)) \subseteq \ell^2\mathbb{Z}$, as v is inert and unramified in K/F, so $\pi_v^{\ell^2}$ is the smallest power of π_v lying in $\operatorname{im}(N_{K/F})$. But L splits A, as $K \subseteq L$, contradicting the choice of L. This proves the claim.

Suppose there is a non-trivial element $aF^{\times\ell} \in \mathcal{N} \cap \mathcal{K}$. Then $[F(\sqrt[\ell]{a}) : F] = \ell$ and $F(\sqrt[\ell]{a}) \subseteq L$. Because $aF^{\times\ell} \in \mathcal{N}$, the symbol algebra $(\omega_\ell, a/F)_\ell$ is split, where ω_ℓ is a primitive ℓth root of unity. Hence, $\omega_\ell \in F^\times$ is a norm from $F(\sqrt[\ell]{a})$. Albert's Theorem (see [A, Theorem 1.1, Chapter IX, §6]) then says that there is a field $K \supseteq F(\sqrt[\ell]{a})$ with K cyclic Galois over F and $[K : F] = \ell^2$. Let v be a valuation on F and w a valuation on K as in Lemma 1.4.

Suppose first that v is ramified in L. Then, by Lemma 1.1(i), there is $\pi \in F^\times$ with $v(\pi) = 1$ and $\sqrt[\ell]{\pi} \in L$. Let $B = (K/F, \sigma, \pi)$. Then $\exp(B) = \ell^2$, just as for A above, and L contains the maximal subfield $F(\sqrt[\ell]{a}, \sqrt[\ell]{\pi})$ of B. So, L splits B, contradicting the choice of L.

Thus, v must be unramified in L. Now, the field $\overline{F}_v$ is finite over k so is finitely generated over its prime field, but is not a finite field. If $\overline{F}_v$ is a global field, there is a cyclic division algebra C over $\overline{F}_v$ of degree and exponent ℓ^2 which is split by $\overline{K}_w$, say $C = (\overline{K}_w/\overline{F}_v, \tau, \widetilde{d})$ for some $\widetilde{d} \in \overline{F}_v^\times$ and a generator τ of $(\overline{K}_w/\overline{F}_v)$. If $\overline{F}_v$ is not a global field, it is still finitely generated over its prime field. Therefore, Lemma 1.4 shows that there is a discrete valuation u on $\overline{F}_v$ with a unique extension to $\overline{K}_w$ such that its residue field $\overline{\overline{K}_w}$ is cyclic Galois of degree ℓ^2 over the u-residue field $\overline{\overline{F}_v}$ of $\overline{F}_v$. Choose any $\widetilde{d} \in \overline{F}_v$ with $u(\widetilde{d}) = 1$, and again let $C = (\overline{K}_w/\overline{F}_v, \tau, \widetilde{d})$. The same argument as for A and B above shows that $\exp(C) = \ell^2$. By Proposition 1.2(ii), $\sqrt[\ell]{\widetilde{d}} \in \overline{L}_{w'}$ for any extension w' of v to L. So, Lemma 1.1(iii) shows that there is $d \in F^\times$ with $\sqrt[\ell]{d} \in L^\times$, $v(d) = 0$, and $\overline{d} = \widetilde{d}$ in $\overline{F}_v$. Let $D = (K/F, \rho, d)$, an algebra of degree ℓ^2 over F. Because D specializes to C with respect to the v-adic valuation on F, we have $\ell^2 = \exp(C) \mid \exp(D)$. But L contains the maximal subfield $F(\sqrt[\ell]{a}, \sqrt[\ell]{d})$ of D. Hence, L splits D, contradicting the choice of L. □

Corollary 1.6. *Let F be a field which is not a global or a finite field, but which is finitely generated over its prime field. Let ℓ be a prime number with $\mu_{\ell^2} \subseteq F^\times$. Then, there is no abelian Galois extension L of F with ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$.*

Proof. First, we will show that ${}_\ell\mathrm{Br}(F) \neq (1)$. For this, let F_0 be a global subfield of F. We assert that the canonical map $\mathrm{res}_{F/F_0} : {}_\ell\mathrm{Br}(F_0) \to {}_\ell\mathrm{Br}(F)$ is non-trivial. Since ${}_\ell\mathrm{Br}(F_0) \neq (1)$ (see [Pi, §18.5, Theorem and Example 5]), this will imply that the image of this map is non-trivial, which implies that ${}_\ell\mathrm{Br}(F) \neq (1)$. We prove our assertion by induction on the transcendence degree $\operatorname{trdeg}(F/F_0)$ of F over F_0. If $\operatorname{trdeg}(F/F_0) = 0$, then F/F_0 is a finite extension. We may assume that F/F_0 is Galois. The structure theorem for $\mathrm{Br}(F)$ shows that if res_{F/F_0} is trivial, then F/F_0 has local degree divisible by ℓ at all but possibly one finite prime of F_0 (see [Pi, loc.cit.]). However, Chebotarev's density theorem shows that the set of finite primes in F_0 which have local degree 1 in F/F_0 has density $1/[F : F_0] > 0$ and it is therefore infinite. This is a contradiction. Assume that we have proved our assertion for $\operatorname{trdeg}(F/F_0) < n$, for some $n \in \mathbb{Z}_{\geqslant 1}$. If $\operatorname{trdeg}(F/F_0) = n$, then let v be a discrete valuation on F trivial on F_0 and whose residue field $\overline{F_v}$ is a finitely generated extension of F_0 satisfying $\operatorname{trdeg}(\overline{F_v}/F_0) < n$. (It is an easy exercise to show that such v exists.) Then, $\mathrm{res}_{\overline{F_v}/F_0}$ can be written as the composition

$$\mathrm{res}_{\overline{F_v}/F_0} : {}_\ell\mathrm{Br}(F_0) \xrightarrow{\mathrm{res}_{F/F_0}} {}_\ell\mathrm{Br}(F) \longrightarrow {}_\ell\mathrm{Br}(\overline{F_v}),$$

where the last map in the composition above is the specialization map. (We remind the reader that in this context the specialization map is the composition of the restriction $\mathrm{res}_{\widehat{F}_v/F} : {}_\ell\mathrm{Br}(F) \to$

${}_\ell\mathrm{Br}(\widehat{F}_v)$ with projection of ${}_\ell\mathrm{Br}(\widehat{F}_v)$ onto the first component in the Witt decomposition of $\mathrm{Br}(\widehat{F}_v)'$ described in the proof of Proposition 1.2(ii).) Since by the induction hypothesis $\mathrm{res}_{\overline{F}_v/F_0}$ is non-trivial, so is res_{F/F_0}. This proves our assertion.

Suppose that there was a field L as in the statement of the corollary. Because $\mu_{\ell^2} \subseteq F^\times$, the $\mathcal{N}$ of the theorem is all of $F^\times/F^{\times\ell}$. The condition $\mathcal{N} \cap \mathcal{K} = (1)$ then forces $\mathcal{K} = (1)$, hence F has no ℓ-Kummer extension in L. Therefore, $[L : F]$ is prime to ℓ; so ${}_\ell\mathrm{Br}(F)$ injects into $\mathrm{Br}(L)$. Since ${}_\ell\mathrm{Br}(F)$ is non-trivial, we have ${}_\ell\mathrm{Br}(F) \neq \mathrm{Br}(L/F)$, a contradiction. □

2. Henselian valued fields

In this section, we study more closely the relationship between the hypothesis $\mu_{\ell^2} \subseteq F^\times$ and the existence of a field extension L/F, such that ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$, in the case where F is a Henselian valued field (i.e. a field endowed with a Henselian valuation v) and ℓ is a prime different from the residual characteristic $\mathrm{char}(\overline{F}_v)$. Although Henselian valued fields are not finitely generated over their prime fields, they occur naturally, for example, as completions of finitely generated fields with respect to any of their discrete valuations. In particular, a local field and a field of iterated power series in finitely many variables with coefficients in any field is a Henselian valued field. Note that if F is a local field and $n \in \mathbb{Z}_{\geqslant 1}$, then ${}_n\mathrm{Br}(F) = \mathrm{Br}(L_n/F)$, where L_n is any extension of degree n of F, in particular the (Galois, cyclic) unramified extension of F of degree n [Se, Chapter XIII, §3, Corollary 1].

Proposition 2.1. *Let ℓ be a prime number, and let k be a field with* $\mathrm{char}(k) \neq \ell$. *Let F be a field with Henselian valuation v with residue field $\overline{F}_v = k$ and value group Γ_F, such that $\mu_{\ell^2} \nsubseteq F^\times$. Let $\{\gamma_i\}_{i\in I} \subseteq \Gamma_F$ map to a $\mathbb{Z}/\ell\mathbb{Z}$ vector space base of $\Gamma_F/\ell\Gamma_F$, and choose $\{\pi_i\}_{i\in I}$ such that $v(\pi_i) = \gamma_i$. Suppose there is a field M algebraic over k such that ${}_\ell\mathrm{Br}(k) = \mathrm{Br}(M/k)$ and no subfield of M which is a cyclic Galois extension of degree ℓ over k lies in a cyclic Galois extension of k of degree ℓ^2. Let M' be the unramified extension of F with $\overline{M'}_v \cong M$, and let $L = M'(\{\sqrt[\ell]{\pi_i}\}_{i\in I})$. Then, ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$.*

Proof. The Henselian valuation v on F yields a direct sum decomposition for the ℓ-primary component $\mathrm{Br}(F)(\ell)$ of $\mathrm{Br}(F)$

$$\mathrm{Br}(F)(\ell) \cong \mathrm{Br}(k)(\ell) \oplus \mathrm{Hom}_c(G_k, \Delta/\Gamma_F)(\ell) \oplus T, \tag{$*$}$$

where G_k is the absolute Galois group of k; $\Delta = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$; Hom_c denotes the group of continuous homomorphisms (where G_k has the profinite group topology and Δ/Γ_F the discrete topology). Since $\mu_{\ell^2} \nsubseteq F^\times$, T has the following description: If $\mu_\ell \subseteq F^\times$, then, after the index set I is given some arbitrary total ordering, T is the $\mathbb{Z}/\ell\mathbb{Z}$-vector space with base consisting of the (Brauer classes of the) ℓ-symbol algebras $(\pi_i, \pi_j/F)_\ell$ for all $i < j$ in I. If $\mu_\ell \nsubseteq F^\times$, then $T = (0)$. See [ASW, §3, Theorem 3.2 and Proposition 3.5] for the decomposition above as well as the description of T. This decomposition of $\mathrm{Br}(F)(\ell)$ is compatible with the scalar extension to L, in that there is a commutative diagram (see [ASW, Proposition 3.3(a)–(b)]):

$$\begin{array}{ccccccc}
\mathrm{Br}(F)(\ell) & \longrightarrow & \mathrm{Br}(k)(\ell) & \oplus & \mathrm{Hom}_c(G_k, \Delta/\Gamma_F)(\ell) & \oplus & T \\
\Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{can}} & & \Big\downarrow{\scriptstyle 0} \\
\mathrm{Br}(L)(\ell) & \longrightarrow & \mathrm{Br}(M)(\ell) & \oplus & \mathrm{Hom}_c(G_M, \Delta/\Gamma_L)(\ell) & \oplus & T'
\end{array}$$

Here, the first and second vertical maps are restriction (i.e., extension of scalars), the third vertical map is the one induced by inclusion $G_M \to G_k$ and the canonical epimorphism $\Delta/\Gamma_F \to \Delta/\Gamma_L$, and the last vertical map is zero, since L splits each generator of T. Note that L is a totally ramified extension of M', with $\Gamma_L = \Gamma_F + \sum_{i\in I} \frac{1}{\ell}\gamma_i$. Hence, $\Gamma_L/\Gamma_F = \frac{1}{\ell}\Gamma_F/\Gamma_F$, which is the ℓ-torsion subgroup of Δ/Γ_F.

For an element $[A] \in \mathrm{Br}(F)(\ell)$ write its components in the direct sum decomposition above as $([B], \chi, [S])$. Suppose that $\exp(A) = \ell$. Then $\exp(B) \mid \ell$; so, M splits B, by hypothesis. Since $\exp(\chi) \mid \ell$, we have $\mathrm{im}(\chi) \subseteq \Gamma_L/\Gamma_F$; so, $\mathrm{can}(\chi) = 0$. The commutative diagram shows that L splits A. Thus, ${}_\ell\mathrm{Br}(F) \subseteq \mathrm{Br}(L/F)$.

For the reverse inclusion, suppose now instead that L splits A. We may assume that $\exp(A) = \ell^2$. The commutative diagram shows that $[B] \in \mathrm{Br}(M/F) = {}_\ell\mathrm{Br}(F)$. Also, $\exp(S) \mid \ell$. Therefore, $\exp(\chi) = \ell^2$. Consider the fixed field K of $\ker(\chi)$. Then, K is an abelian Galois field extension of k of exponent ℓ^2. Let χ' be the image of χ in $\mathrm{Hom}(G_k, \Delta/\Gamma_L)$, and let N be the fixed field of $\ker(\chi')$. Then, $k \subseteq N \subseteq K$ and N is the smallest subfield of K containing k such that $\exp(\mathcal{G}(K/N)) = \ell$. Because $\mathrm{can}(\chi) = 0$, the restriction of χ' to G_M is trivial; that is, $M \cdot N = M$, so $N \subseteq M$. Because $\mathcal{G}(K/k)$ has exponent ℓ^2 there is a field K_0 with $k \subseteq K_0 \subseteq K$ and K_0 cyclic of degree ℓ^2 over k. Let $N_0 = K_0 \cap N$, which is the subfield of K_0 of degree ℓ over k. We have $N_0 \subseteq N \subseteq M$ and N_0 lies in the cyclic extension K_0 of degree ℓ^2 over k. This contradicts the hypothesis on M. Thus, $\mathrm{Br}(L/F) \subseteq {}_\ell\mathrm{Br}(F)$, completing the proof. □

Proposition 2.2. *Let ℓ be a prime number, and let F be a field with Henselian valuation v, with residue field $\overline{F}_v$ and value group Γ_F. Suppose $\overline{F}_v$ is a finite field with* $\mathrm{char}(\overline{F}_v) \neq \ell$ *and* $\dim_{\mathbb{Z}/\ell\mathbb{Z}}(\Gamma_F/\ell\Gamma_F) \geqslant 2$.

(i) *If $\mu_{\ell^2} \subseteq F^\times$, then ${}_\ell\mathrm{Br}(F) \neq \mathrm{Br}(L/F)$ for any field L algebraic over F.*
(ii) *If $\mu_{\ell^2} \nsubseteq F^\times$, then ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$ for some abelian exponent ℓ Galois extension L of F.*

Proof. (i) Since v is Henselian, we have the direct sum decomposition of $\mathrm{Br}(F)(\ell)$ as in $(*)$, where $\overline{F}_v := k$ and T is generated by certain totally ramified symbol algebras of exponent ℓ^r, where r is maximal such that $\mu_{\ell^r} \subseteq F^\times$. Of course, $\mathrm{Br}(k) = (0)$, as k is finite. Suppose there was a field L algebraic over F with ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$. Since v is Henselian, v has a unique extension to L, which we again call v, and v on L is also Henselian. So, there is a decomposition of $\mathrm{Br}(L)(\ell)$ like $(*)$ for $\mathrm{Br}(F)(\ell)$. Now, take any $\pi, \rho \in F^\times$ such that $v(\pi)$ and $v(\rho)$ are $\mathbb{Z}/\ell\mathbb{Z}$-linearly independent in $\Gamma_F/\ell\Gamma_F$, and let $A := (\pi, \rho/F)_\ell$. Since v is indecomposed in $F(\sqrt[\ell]{\pi})/F$, we have $v(\mathrm{im}(N_{F(\sqrt[\ell]{\pi})/F})) \subseteq \ell\Gamma_{F(\sqrt[\ell]{\pi})} = \langle v(\pi)\rangle + \ell\Gamma_F$. This contains $v(\rho^j)$ if and only if $\ell \mid j$. Hence $\exp(A) = \ell$. Since L splits A, $v(\pi)$ and $v(\rho)$ must be $\mathbb{Z}/\ell\mathbb{Z}$-linearly dependent in $\Gamma_L/\ell\Gamma_L$. For, otherwise the same argument as over F would show that $A \otimes_F L$ has exponent ℓ. Thus, there is $s \in F$ with $v(s) \in \ell\Gamma_L$ but $v(s) \notin \ell\Gamma_F$. Write $s = uy^\ell$ with $u, y \in L^\times$ and $v(u) = 0$.

Now, let us choose any $\widetilde{a} \in k^\times \setminus k^{\times\ell}$, and any $a \in F^\times$ with $v(a) = 0$ and $\bar{a} = \widetilde{a}$. Let $B := (a, s/F)_{\ell^2}$. Because v is inert (i.e. indecomposed and unramified) in $F(\sqrt[\ell^2]{a})/F$, we have $v(\mathrm{im}(N_{F(\sqrt[\ell^2]{a})/F})) \subseteq \ell^2\Gamma_F$. But, since $v(s) \notin \ell\Gamma_F$, we have $\ell v(s) \notin \ell^2\Gamma_F$, hence, $\exp(B) = \ell^2$.

Suppose that $\sqrt[\ell]{\widetilde{a}} \in \overline{L}_v$. Then, $\sqrt[\ell]{a} \in L$, by Hensel's Lemma. So, in $\mathrm{Br}(L)(\ell)$ we have $B \otimes_F L \sim (\sqrt[\ell]{a}, uy^\ell/L)_\ell \sim (\sqrt[\ell]{a}, u/L)_\ell$. In the isomorphism like $(*)$ for $\mathrm{Br}(L)(\ell)$, $(\sqrt[\ell]{a}, u/L)_\ell$ has image $(\sqrt[\ell]{\widetilde{a}}, \bar{u}/\overline{L}_v)_\ell$ in $\mathrm{Br}(\overline{L}_v)$. However, since $\overline{L}_v$ is algebraic over a finite field, $\mathrm{Br}(\overline{L}_v) = (0)$.

Hence L splits B, which contradicts the hypothesis on L. Therefore, $\sqrt[\ell]{\bar{a}} \notin \overline{L}_v$; hence, $\overline{L}_v(\sqrt[\ell]{\bar{a}})$ is the unique extension of $\overline{L}_v$ of degree ℓ.

We claim: For any $x \in L^\times$, if $v(x) \in \ell\Gamma_L$, then $x = a^i c^\ell$ for some $i \in \mathbb{Z}$ and $c \in L^\times$. Indeed, $x = bd^\ell$ for some $b \in L^\times$ with $v(b) = 0$. From Kummer theory, we have $\bar{b} = \bar{a}^i \bar{y}^\ell$ for some $y \in L^\times$ with $v(y) = 0$. Then $b/a^i y^\ell = 1 + m$ for some $m \in L$ with $v(m) > 0$ (or $m = 0$). By Hensel's Lemma, $1 + m = z^\ell$ for some $z \in L^\times$. Thus, $x = a^i(yzd)^\ell$, as claimed.

Now, let $C := (a, \pi/F)_\ell$. Since v is inert in $F(\sqrt[\ell]{a})/F$, and $v(\pi) \notin \ell\Gamma_F$, we have $\exp(C) = \ell$. Hence L splits C. So, as v is inert in $L(\sqrt[\ell]{a})/L$, we must have $v(\pi) \in v(\mathrm{im}(N_{L(\sqrt[\ell]{a})/L})) \subseteq \ell\Gamma_L$. The claim above shows that $\pi = a^i c^\ell$ for some $c \in L^\times$. The same argument as for π shows that $\rho = a^j e^\ell$ for some $e \in L^\times$. Let $D = (\pi a^{-i}, \rho a^{-j}/F)_{\ell^2}$. Since $v(\pi a^{-i}) = v(\pi)$ and $v(\rho a^{-j}) = v(\rho)$, the same argument as for A above shows that $\exp(D) = \ell^2$. However, $D \otimes_F L \sim (c^\ell, e^\ell/L)_{\ell^2}$, which is clearly trivial in $\mathrm{Br}(L)$. Therefore D is split by L, which contradicts the choice of L.

(ii) This is a direct consequence of Proposition 2.1. Indeed, since $k := \overline{F}_v$ is finite, we have ${}_\ell\mathrm{Br}(k) = \mathrm{Br}(k) = 0$, so in the hypotheses of Proposition 2.1 we may take M/k to be the trivial extension. □

Corollary 2.3. *Let $F = k((t))$, where k is a local field. Let ℓ be a prime number with* $\mathrm{char}(\bar{k}) \neq \ell$.

(i) *If $\mu_{\ell^2} \subseteq F^\times$, then ${}_\ell\mathrm{Br}(F) \neq \mathrm{Br}(L/F)$ for any field L algebraic over F.*
(ii) *If $\mu_{\ell^2} \nsubseteq F^\times$, then ${}_\ell\mathrm{Br}(F) = \mathrm{Br}(L/F)$ for some abelian Galois extension L of F with ℓ-torsion Galois group.*

Proof. Let w be the usual complete discrete, hence Henselian, valuation on k. We use here the valuation v on F given by $v(\sum_{i \geqslant N} c_i t^i) = (w(c_j), j)$, where j is minimal such that $c_j \neq 0$. Then, $\overline{F}_v = \bar{k}_w$, which is a finite field, and $\Gamma_F = \mathbb{Z} \times \mathbb{Z}$, which has ℓ-rank 2. Note that v is the composite valuation built from the complete discrete (so Henselian) t-adic valuation u on F (given by $u(\sum_{i \geqslant N} c_i t^i) = \min\{j \mid c_j \neq 0\}$) and the valuation w on the residue field $\overline{F}_u = k$, cf. [B, Chapter VI, §4.1]. Because u and w are each Henselian, v is also Henselian, by [EP, p. 90, Corollary 4.1.4]. With this v, Corollary 2.3 follows immediately from Proposition 2.2. □

Remark. Corollary 2.3(i) in the case $\ell = 2$ is essentially the example discussed in [AS].

3. A concrete example

In this section we show that the Henselian valued field $F := \mathbb{Q}((t))$ (rank one discrete valuation of uniformizer t and residue field $\mathbb{Q}$) satisfies the hypotheses of Proposition 2.1, for all prime numbers ℓ. This will allow us to construct explicit Galois extensions $L_\ell/\mathbb{Q}((t))$ for which ${}_\ell\mathrm{Br}(\mathbb{Q}((t))) = \mathrm{Br}(L_\ell/\mathbb{Q}((t)))$, for all primes ℓ.

The following is a refinement of the main result in [KS1] for base-field $\mathbb{Q}$.

Proposition 3.1. *Let ℓ be a prime number. There exists an abelian Galois extension L of $\mathbb{Q}$ of exponent ℓ such that*

(i) ${}_\ell\mathrm{Br}(\mathbb{Q}) = \mathrm{Br}(L/F)$; *and*
(ii) *no cyclic subextension of $L/\mathbb{Q}$ of degree ℓ lies in a cyclic Galois extension of $\mathbb{Q}$ of degree ℓ^2.*

Proof. Recall the following fact, which we will use frequently in the proof: Let $\mathbb{Q} \subseteq M \subseteq K$ be fields with K Galois over $\mathbb{Q}$ and $[K : \mathbb{Q}] < \infty$. Let p be any prime number. Then, p splits completely in M if and only if p splits completely in the normal closure of M over $\mathbb{Q}$. In order to see this, let P be a prime of K lying over p, and let D_P be the decomposition field of P over p. Now, take into account that for any $\sigma \in \mathcal{G}(K/\mathbb{Q})$, the decomposition field D_{P^σ} of P^σ over p satisfies $D_{P^\sigma} = \sigma(D_P)$ and that p splits completely in M if and only if $M \subseteq D_{P^\sigma}$, for all $\sigma \in \mathcal{G}(K/\mathbb{Q})$ (see [Mar, Chapter 4, p. 108 and Theorem 29(i), p. 104]).

Case I. Assume ℓ is odd.

For any prime number p with $p \equiv 1 \pmod{\ell}$ let $L^{(p)}$ denote the unique subfield of $\mathbb{Q}(\mu_p)$ with $[L^{(p)} : \mathbb{Q}] = \ell$. We will need the following "Reciprocity Lemma:"

Lemma 3.2. *Let p and q be distinct prime numbers with $p \equiv 1 \pmod{\ell}$. Then, q splits completely in $L^{(p)}$ iff p splits completely in $\mathbb{Q}(\sqrt[\ell]{q})$.*

Proof. q splits completely in $L^{(p)} \Leftrightarrow [\mathbb{F}_q(\mu_p) : \mathbb{F}_q] \mid (p-1)/\ell \Leftrightarrow$ the order of q in $(\mathbb{Z}/p\mathbb{Z})^\times$ divides $(p-1)/\ell \Leftrightarrow q$ is an ℓth power in $(\mathbb{Z}/p\mathbb{Z})^\times \Leftrightarrow x^\ell - q$ has a root in $\mathbb{Z}/p\mathbb{Z} \Leftrightarrow x^\ell - q$ factors into linear factors in $\mathbb{Z}/p\mathbb{Z}[x] \Leftrightarrow p$ splits completely in $\mathbb{Q}(\sqrt[\ell]{q})$. □

We return to the proof of Proposition 3.1.

Step 1. We claim: There is a prime number p_1 satisfying

(a) $p_1 \equiv 1 \pmod{\ell}$ (i.e., p_1 splits completely in $\mathbb{Q}(\mu_\ell)$);
(b) $p_1 \not\equiv 1 \pmod{\ell^2}$ (i.e., p_1 does not split completely in $\mathbb{Q}(\mu_{\ell^2})$);
(c) ℓ is inert in $L^{(p_1)}$.

Note that by the Reciprocity Lemma condition (c) is equivalent to the condition that p_1 does not split completely in $\mathbb{Q}(\sqrt[\ell]{\ell})$. By the fact noted at the beginning of the proof, this is equivalent to

(c′) p_1 does not split completely in $\mathbb{Q}(\mu_\ell, \sqrt[\ell]{\ell})$.

We show that these "Chebotarev conditions" are compatible. Consider the field $K = \mathbb{Q}(\mu_{\ell^2}, \sqrt[\ell]{l})$. We have $\mathbb{Q}(\mu_{\ell^2}) \neq \mathbb{Q}(\mu_\ell, \sqrt[\ell]{\ell})$, since the second field is not abelian Galois over $\mathbb{Q}$. Therefore, $\mathcal{G}(K/\mathbb{Q}(\mu_\ell)) \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$. Choose an element σ of order ℓ in this group which lies neither in $\mathcal{G}(K/\mathbb{Q}(\mu_{\ell^2}))$ nor in $\mathcal{G}(K/\mathbb{Q}(\mu_\ell, \sqrt[\ell]{l}))$. Let p_1 be any prime number whose Frobenius class is that of σ. Then, there is a prime P of K lying over p_1 such that the fixed field D of σ is the decomposition field of P over p_1. By the fact noted at the beginning of the proof, p_1 satisfies conditions (a), (b), and (c′). Set $L_1 = L^{(p_1)}$. Suppose there was a cyclic Galois extension M of $\mathbb{Q}$ of degree ℓ^2 with $L_1 \subseteq M$. Because $L_1 \cdot \mathbb{Q}_{p_1}$ is totally and tamely ramified over $\mathbb{Q}_{p_1}$ of degree ℓ, $M \cdot \mathbb{Q}_{p_1}$ must be cyclic of degree ℓ^2 over $\mathbb{Q}_{p_1}$, and is necessarily totally ramified over $\mathbb{Q}_{p_1}$. But, this cannot occur, as $\mu_{\ell^2} \not\subseteq \mathbb{Q}_{p_1}$. So, there is no such M.

Step 2. Let q_2 be a prime number different from p_1 and from ℓ. We claim: There exists a prime number p_2 satisfying

(a$_2$) $p_2 \equiv 1 \pmod{\ell}$ (i.e., p_2 splits completely in $\mathbb{Q}(\mu_\ell)$);
(b$_2$) $p_2 \not\equiv 1 \pmod{\ell^2}$ (i.e., p_2 does not split completely in $\mathbb{Q}(\mu_{\ell^2})$);
(c$_2$) p_2 splits completely in L_1;

(d_2) p_1 splits completely in $L^{(p_2)}$ (which is equivalent, by the Reciprocity Lemma to: p_2 splits completely in $\mathbb{Q}(\sqrt[\ell]{p_1})$);
(e_2) q_2 is inert in $L^{(p_2)}$ (which is equivalent to: p_2 does not split completely in $\mathbb{Q}(\sqrt[\ell]{q_2})$).

Let $K_2 = L_1(\mu_\ell, \sqrt[\ell]{p_1})$. Since ℓ does not ramify in $L_1(\sqrt[\ell]{p_1})$, we have $\mu_{\ell^2} \not\subseteq K_2^\times$; likewise, since q_2 does not ramify in $K_2(\mu_{\ell^2})$, we have $\sqrt[\ell]{q_2} \notin K_2(\mu_{\ell^2})$. Consequently, the field $N_2 = K_2(\mu_{\ell^2}, \sqrt[\ell]{q_2})$ is abelian noncyclic Galois over K_2 of degree ℓ^2. Choose any element σ_2 of order ℓ in $\mathcal{G}(N_2/K_2)$ not lying in $\mathcal{G}(N_2/K_2(\mu_\ell^2))$ nor in $\mathcal{G}(N_2/K_2(\sqrt[\ell]{q_2}))$. Let p_2 be any prime whose Frobenius class is that of σ_2. Then, p_2 satisfies conditions (a_2)–(e_2). (p_2 splits completely in K_2 but not in $K_2(\mu_{\ell^2})$, so it cannot split completely in $\mathbb{Q}(\mu_{\ell^2})$; likewise, p_2 cannot split completely in $\mathbb{Q}(\sqrt[\ell]{q_2})$.) Set $L_2 = L^{(p_2)}$.

Continue in this fashion:

Step j. Let q_j be a prime number different from $\ell, p_1, \ldots, p_{j-1}, q_2, \ldots, q_{j-1}$. We next find a prime number p_j different from the primes we already have, satisfying

(a_j) $p_j \equiv 1 \pmod{\ell}$ (i.e., p_j splits completely in $\mathbb{Q}(\mu_\ell)$);
(b_j) $p_j \not\equiv 1 \pmod{\ell^2}$ (i.e., p_j does not split completely in $\mathbb{Q}(\mu_{\ell^2})$);
(c_j) p_j splits completely in $L_1 \cdot L_2 \cdot \ldots \cdot L_{j-1}$;
(d_j) $p_1, \ldots, p_{j-1}$ each split completely in $L^{(p_j)}$ (which is equivalent to: p_j splits completely in $\mathbb{Q}(\sqrt[\ell]{p_1}, \ldots, \sqrt[\ell]{p_{j-1}})$);
(e_j) q_j is inert in $L^{(p_j)}$ (which is equivalent to: p_j is not split completely in $\mathbb{Q}(\sqrt[\ell]{q_j})$).

Let $K_j = L_1 \cdots L_{j-1}(\mu_\ell, \sqrt[\ell]{p_1}, \ldots, \sqrt[\ell]{p_{j-1}})$. Since ℓ does not ramify in the field $L_1 \cdots L_{j-1}(\sqrt[\ell]{p_1}, \ldots, \sqrt[\ell]{p_{j-1}})$, we have $\mu_{\ell^2} \not\subseteq K_j^\times$; likewise, since q_j is unramified in $K_j(\mu_{\ell^2})$, we have $\sqrt[\ell]{q_j} \notin K_j(\mu_{\ell^2})$. Consequently, the field $N_j = K_j(\mu_{\ell^2}, \sqrt[\ell]{q_j})$ is non-cyclic abelian of degree ℓ^2 over K_j. Choose any element σ_j of order ℓ in $\mathcal{G}(N_j/K_j)$ not lying in $\mathcal{G}(N_j/K_j(\mu_\ell^2))$ nor in $\mathcal{G}(N_j/K_j(\sqrt[\ell]{q_j}))$. Let p_j be any prime whose Frobenius class is that of σ_j. Then, p_j satisfies conditions (a_j)–(e_j). Set $L_j = L^{(p_j)}$. Let $S_i = L_1 L_2 \cdots L_{i-1} L_{i+1} \cdots L_j$ for $1 \leqslant i \leqslant j$. Take any cyclic Galois extension K of $\mathbb{Q}$ with $K \subseteq L_1 \cdots L_j$. Then, $K \not\subseteq S_i$ for some i, since $S_1 \cap \cdots \cap S_j = \mathbb{Q}$; so $K \cdot S_i = L_1 \cdots L_j$. Because p_i splits completely in S_i but is totally ramified in L_i, and hence in $L_1 \cdots L_j$, this p_i must be totally ramified in K. Therefore, the same argument as in step 1 shows that K does not embed in any cyclic Galois extension of $\mathbb{Q}$ of degree ℓ^2.

By continuing this process, we obtain two sequences of distinct prime numbers $\{p_1, p_2, \ldots\}$, $\{q_2, q_3, \ldots\}$ satisfying: $p_i \equiv 1 \pmod{\ell}$ for all i; $p_i \not\equiv 1 \pmod{\ell^2}$ for all i; p_i splits completely in $L_j = L^{(p_j)}$ for all $j \neq i$; q_j is inert in L_j for all $j \geqslant 2$. Clearly the q_j can be chosen so that $\{\ell, p_1, p_2, \ldots, q_2, q_3, \ldots\}$ is the set of all prime numbers. We claim that $L = L_1 L_2 \cdots$ has local degree ℓ at all the finite primes of $\mathbb{Q}$. At p_i this is true because p_i ramifies in L_i and splits completely in L_j for all $j \neq i$. At ℓ and at each q_i, $L/\mathbb{Q}$ is unramified of exponent ℓ, hence locally of degree ℓ, since ℓ is inert in L_1 and q_i is inert in L_i. It now follows from the fundamental theorem for the Brauer group of a global field that L splits exactly the ℓ-torsion of $\mathrm{Br}(\mathbb{Q})$. Note that no cyclic Galois extension K of $\mathbb{Q}$ lying in L embeds in a cyclic Galois extension of $\mathbb{Q}$ of degree ℓ^2, since we saw in step j that this is true for subfields of each $L_1 \cdots L_j$. Thus, our L has the required properties, completing Case I.

Case II. Assume now that $\ell = 2$.

Step 1. Choose prime numbers p_1 and p_2 with $p_1 \equiv 1 \pmod 8$ and $p_2 \equiv 3 \pmod 8$, and set $L_1 = \mathbb{Q}(\sqrt{-p_1 p_2})$. Note that 2 is inert in L_1, as $-p_1 p_2 \equiv -3 \pmod 8$.

Step 2. Let q_2 be a prime number different from 2, p_1, and p_2. Let $L_2 = \mathbb{Q}(\sqrt{p_3 p_4})$, where prime numbers p_3 and p_4 are chosen different from 2, p_1 and p_2 and are required to satisfy:

(a) $p_3 \equiv p_4 \equiv 3 \pmod 8$ (so 2 splits in L_2);
(b) p_3 and p_4 split in L_1;
(c) p_1 and p_2 split in L_2;
(d) q_2 is inert in L_2.

To assure that these conditions can hold, we choose p_3 satisfying $p_3 \equiv 3 \pmod 8$, and p_3 is inert in $\mathbb{Q}(\sqrt{p_1})$, split in $\mathbb{Q}(\sqrt{p_2})$, and inert in $\mathbb{Q}(\sqrt{q_2})$. Choose p_4 satisfying $p_4 \equiv 3 \pmod 8$ and p_4 is inert in $\mathbb{Q}(\sqrt{p_1})$, split in $\mathbb{Q}(\sqrt{p_2})$, and split in $\mathbb{Q}(\sqrt{q_2})$. Note that the condition $p_3 \equiv 3 \pmod 8$ is equivalent to: The Frobenius automorphism for p_3 in $\mathcal{G}(\mathbb{Q}(\mu_8)/\mathbb{Q})$ has fixed field $\mathbb{Q}(\sqrt{-2})$. To find a suitable p_3, let $K_3 = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2})$ and $N_3 = K_3(\mu_8, \sqrt{p_1}, \sqrt{q_2})$. So, N_3 is abelian Galois over K_3 of degree 8 and exponent 2. Choose the $\sigma \in \mathcal{G}(N_3/K_3)$ with $\sigma(\sqrt{-1}) = -\sqrt{-1}$, $\sigma(\sqrt{p_1}) = -\sqrt{p_1}$, and $\sigma(\sqrt{q_2}) = -\sqrt{q_2}$; choose p_3 with Frobenius σ for $N_3/\mathbb{Q}$. The argument for p_4 is analogous, with $K_4 = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2}, \sqrt{q_2})$ and $N_4 = N_3$. These conditions yield the following in terms of Legendre symbols: $(\frac{-1}{p_3}) = (\frac{p_1}{p_3}) = -1$, $(\frac{p_2}{p_3}) = 1$, and $(\frac{q_2}{p_3}) = -1$; $(\frac{-1}{p_4}) = (\frac{p_1}{p_4}) = -1$, $(\frac{p_2}{p_4}) = (\frac{q_2}{p_4}) = 1$. Hence, $(\frac{-p_1 p_2}{p_3}) = (\frac{-p_1 p_2}{p_4}) = 1$, which verifies condition (b). Also, $(\frac{p_3 p_4}{p_1}) = (\frac{p_3}{p_1})(\frac{p_4}{p_1}) = (\frac{p_1}{p_3})(\frac{p_1}{p_4}) = 1$ and $(\frac{p_3 p_4}{p_2}) = (\frac{p_3}{p_2})(\frac{p_4}{p_2}) = (-1)^2(\frac{p_2}{p_3})(\frac{p_2}{p_4}) = 1$, showing that condition (c) holds. Further, $(\frac{p_3 p_4}{q_2}) = (\frac{p_3}{q_2})(\frac{p_4}{q_2}) = (-1)^{q_2-1}(\frac{q_2}{p_3})(\frac{q_2}{p_4}) = -1$, verifying condition (d). Thus, we do have p_3 and p_4 with the specified properties.

Continue in this fashion.

Step j. At this point, we have chosen primes $p_1, \dots, p_{2j-2}$ and $q_2, \dots, q_{j-1}$. Let q_j be an odd prime number different from any of these primes. Let $L_j = \mathbb{Q}(\sqrt{p_{2j-1} p_{2j}})$, where p_{2j-1} and p_{2j} are distinct odd primes different from any of the p_i and q_i we already have, and satisfying the conditions:

(a_j) $p_{2j-1} \equiv p_{2j} \equiv 3 \pmod 8$ (so 2 splits in L_j);
(b_j) p_{2j-1} and p_{2j} split completely in $L_1, L_2, \dots, L_{j-1}$;
(c_j) $p_1, p_2, \dots, p_{2j-2}$ each split in L_j;
(d_j) q_j is inert in L_j.

To assure that these conditions can be satisfied, we choose p_{2j-1} so that $p_{2j-1} \equiv 3 \pmod 8$, and p_{2j-1} is inert in $\mathbb{Q}(\sqrt{p_1})$ but split in each of $\mathbb{Q}(\sqrt{p_2}), \dots, \mathbb{Q}(\sqrt{p_{2j-2}})$, and inert in $\mathbb{Q}(\sqrt{q_j})$; we choose p_{2j} so that $p_{2j} \equiv 3 \pmod 8$, and p_{2j} is inert in $\mathbb{Q}(\sqrt{p_1})$ but split in each of $\mathbb{Q}(\sqrt{p_2}), \dots, \mathbb{Q}(\sqrt{p_{2j-2}})$, and split in $\mathbb{Q}(\sqrt{q_j})$. The Chebotarev density theorem assures the existence of such p_{2j-1} and p_{2j}. (For p_{2j-1}, let $K_{2j-1} = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2}, \dots, \sqrt{p_{2j-2}})$ and $N_{2j-1} = K_{2j-1}(\mu_8, \sqrt{p_1}, \sqrt{q_j})$, and argue as in step 2. The case of p_{2j} is handled analogously.) Routine calculations with Legendre symbols as in step 2 show that these primes satisfy conditions (a_j)–(d_j).

Now let $L = L_1 L_2 \cdots$. Arguments like those in Case I show that we can choose the q_j so that the p_i and q_j are all the prime numbers. Again as in Case I, we find that L has local degree 2 at each prime number; it also has local degree 2 at ∞, since this is the case for L_1. Furthermore, since p_{2j} is totally ramified of degree 2 in L_j but split in the L_i for $i \neq j$, and $\mu_4 \not\subseteq L$ the argument of Case I applies to show that no quadratic extension of $\mathbb{Q}$ within L lies in a cyclic Galois extension of $\mathbb{Q}$ of degree 4. $\square$

Corollary 3.3. ${}_2\mathrm{Br}(\mathbb{Q}((t))) = \mathrm{Br}(L_0((\sqrt{t}))/\mathbb{Q}((t)))$ *for a suitable* 2*-Kummer extension* L_0 *of* $\mathbb{Q}$.

Corollary 3.4. *For any odd prime number* ℓ, ${}_\ell\mathrm{Br}(\mathbb{Q}((t))) = \mathrm{Br}(L_0((\sqrt[\ell]{t}))/\mathbb{Q}((t)))$ *for a suitable exponent* ℓ *abelian Galois extension* L_0 *of* $\mathbb{Q}$.

Corollaries 3.3 and 3.4 follow immediately from Propositions 2.1 and 3.1.

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